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Remarks on the degree of instability☆

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ABSTRACT

Linear systems of differential equations allowing of functions in quadratic forms that do not increase along trajectories with time are considered. The relations between the indices of inertia of these forms and the degrees of instability of equilibrium states are indicated. These assertions generalize known results from the oscillation theory of linear systems with dissipation, and reveal the mechanism of loss of stability when non-increasing quadratic forms lose the property of a minimum.

Linear equations of motion of systems with gyroscopic forces and energy dissipation have the form

$$K\ddot{x} + (\Gamma + D)\dot{x} + Px = 0, \quad x \in \mathbb{R}^k \tag{1}$$

where K is the inertia operator ($K^* = K$ and $K > 0$), Γ is the skew-symmetric operator of gyroscopic forces, the symmetric operator P determines the potential energy of the system $V(x) = (Px, x)/2$, and $D^* = D$, $D \geq 0$, sets the dissipative Raleigh function $\Phi(x) = (D\dot{x}/\dot{x})/2 \geq 0$. From Eq. (1) we have the law of change in total energy

$$(T + V)^{\cdot} = -2\Phi \tag{2}$$

where $T = (k\dot{x}/\dot{x})/2$ is the kinetic energy of the system.

Let $\det P \neq 0$. Then $x = 0$ is an isolated equilibrium position of system (1). Its degree of Poincaré instability p is a negative index of inertia of quadratic form V. As quadratic form T is positive definite, p is identical with the index of inertia of the total energy $T + V$.

The *degree of instability*, u, of the state of equilibrium $x = 0$, $\dot{x} = 0$ will be the name given to the number of roots of the characteristic equation

$$\det(\lambda^2 K + \lambda(\Gamma + D) + P) = 0$$

in the right-hand complex half-plane (taking into account their ratios). According to Kelvin's classical theorem

$$u \equiv p(\mathrm{mod}\,2) \tag{3}$$

On the other hand, as has been shown,[1] if $D > 0$ (dissipation is total), then $u = p$. The following inequality has been proved[2]

$$u \leq p \tag{4}$$

which holds for systems with partial energy dissipation ($D \geq 0$). This inequality is also valid in the more general case where the kinetic energy is non-degenerate but not necessarily positive definite. Then, in inequality (4) p has the meaning of the negative index of inertia of the quadratic form $T + V$.

The aim of the present paper is to extend these results to systems of the most general type.

Consider a linear autonomous system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \tag{5}$$

with non-degenerate operator A. In particular, the equilibrium $x = 0$ will be isolated.

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We will assume that system (5) allows of the quadratic form

$$F(x) = (Bx, x)/2 \tag{6}$$

such that $\dot{F} \leq 0$. In other words

$$BA + A^*B = -S \tag{7}$$

and here $S^* = S$ and $S \geq 0$. Form (6) is assumed to be non-degenerate: $\det B \neq 0$.

Note that the second-order system (1) can be represented in the form (5). Here, $n = 2k$ and the function F stands for the total energy.

Suppose $i^{\pm}$ are indices of inertia of the quadratic form (6), u is the degree of instability of equilibrium $x = 0$, that is the number of eigenvalues of the operator A in the right-hand complex half-plane. It has been proved[3] that

$$u \equiv i^{-} (\operatorname{mod} 2)$$

in this most general case. For system (1), $i^{-} = p$.

Theorem 1. *The inequality $u \leq i^{-}$ holds.*

This theorem obviously contains, as a special case, the result obtained by Wimmer[2] for equations of dynamics with dissipation. In the conservative case (when $S = 0$), the inequality $u \leq i^{+}$ also holds.

Theorem 2. *If the pierced cone.*

$$\{x \in \mathbb{R}^n : (Sx, x) = 0\} \setminus \{x = 0\} \tag{8}$$

contains no closed trajectories of system (5), then $u = i^{-}$.

The condition of the theorem resembles the Barbashin–Krasovskii condition in well-known theorems of asymptotic stability and instability. In particular, it is certainly satisfied for system (1) with complete energy dissipation ($D > 0$). Therefore, Theorem 2 contains, as a special case, a result obtained by Zajac[1] that holds not only for the positive definite quadratic form T.

If $S > 0$, then set (8) is empty. In this case, Theorem 2 becomes into the Ostrowski–Schneider theorem,[4] from which, in turn, the conclusions of Theorems 1 and 2 are derived using special limit transitions.

Proof of Theorems 1 and 2. We will replace system (5) with the following system

$$\dot{x} = Ax - \mu Bx \tag{9}$$

where μ is a non-negative real parameter. It is clear that

$$(Bx, x)^{\cdot}/2 = -\mu(Bx, Bx) \tag{10}$$

i.e., when $\mu > 0$ this derivative is negative definite. Consequently, according to the Ostrowski–Schneider theorem, the number of eigenvalues of the operator $A - \mu B$ in the right-hand complex half-plane is equal to i^{-}. The remainder may lie in the left-hand half-plane or on the imaginary axis. We will show that this possibility is not in fact realized. If this is not so, linear system (9) has a non-trivial τ-periodic solution $t \mapsto z(t)$. We substitute this solution into inequality (10) and average both sides over the interval $[0, \tau]$:

$$0 = \frac{1}{\tau}\int_0^{\tau} (Bz(t), z(t))^{\cdot} dt \leq -\frac{\mu}{\tau}\int_0^{\tau} (Bz(t), Bz(t)) dt$$

As the operator B is non-degenerate, and $z(t) \not\equiv 0$, the integral on the right is positive. We arrive at a contradiction.

We now let the parameter μ tend to zero. The eigenvalues of the operator $A - \mu B$ can be numbered such that they depend continuously on μ. Note, by the way, that in the case of multiple eigenvalues the dependence on the parameter can be non-smooth. However, here only the property of continuity is sufficient, Because, for all small $\mu > 0$, the eigenvalues of the operator $A - \mu B$ do not lie on the imaginary axis, when $\mu = 0$ the number of eigenvalues to the left and right of this axis can only decrease: some of them can appear when $\mu = 0$ on the imaginary axis, which proves Theorem 1.

We will now show that, when the condition of Theorem 2 is satisfied, none of the eigenvalues of the operator A lies on the imaginary axis. If this observation is taken into account, the conclusion of Theorem 2 follows, of course, on taking the limit as $\mu \to 0$.

If operator A has pure imaginary eigenvalues, the linear equation (5) allows of a non-trivial τ-periodic solution $z(\cdot)$. Because

$$(Bx, x)^{\cdot} = -(Sx, x)$$

and the function $t \mapsto z(t)$ is periodic, it follows that

$$0 = \int_0^{\tau} (Bz(t), z(t))^{\cdot} dt = -\int_0^{\tau} (Sz(t), z(t)) dt \tag{11}$$

The integrand on the right is continuous and non-negative. This means that the closed curve $z(\cdot)$ lies wholly in the cone

$$\{x \in R^n : (Sx, x) = 0\}$$

In fact, it lies wholly in the pierced cone (8): if $z(t) = 0$ for a certain t, then $z(t) \equiv 0$ according to the uniqueness theorem. Thus, set (8) contains the entire closed trajectory of system (5). However, this contradicts the condition of Theorem 2.

Example. Consider the second-order linear system

$$\ddot{x} = kx + \Gamma x, \quad x \in \mathbb{R}^n \tag{12}$$

Here k is a real number (so that the first term on the right represents the central force), and Γ is a non-degenerate skew-symmetric operator (in particular, n is even). The second term on the right is the additional positional force acting on the system. Since $\Gamma^* = -\Gamma$ and $\Gamma \neq 0$, this force is certainly not a potential force. We will assume that

$$F = (\Gamma\dot{x}, x) \tag{13}$$

The function F acts as the angular momentum of the multidimensional dynamical system. Then

$$\dot{F} = -(\Gamma x, \Gamma x) \le 0$$

Since $\det\Gamma \neq 0$, the quadratic form (13) is non-degenerate in the variables x and $\dot{x}$, and its index of inertia is equal to n. Further, $\dot{F} = 0$ only when $x = 0$. However, the condition of Theorem 2 will then be satisfied, and consequently the degree of instability of the state of equilibrium $x = 0, \dot{x} = 0$ is equal to n (irrespective of the sign of the coefficient of elasticity k). Nevertheless, this result can also be obtained by direct calculations of the eigenvalues.

In conclusion, we return anew to Eq. (1) and consider the case where $P < 0$. In particular, the degree of Poincaré instability p is identical with the number of degrees of freedom k. We will assume

$$F = -(x, \dot{x}) - (Dx, x)/2. \tag{14}$$

It is simple to check that the derivative of this function, by virtue of system (1), is equal to

$$-(\dot{x}, \dot{x}) + (\Gamma\dot{x}, x) + (Px, x)$$

Since

$$2(\Gamma\dot{x}, x) = -2(\dot{x}, \Gamma x) \le (\dot{x}, \dot{x}) + (\Gamma x, \Gamma x)$$

it follows that

$$\dot{F} \le -(\dot{x}, \dot{x})/2 + (Px, x) + (\Gamma x, \Gamma x)$$

If

$$P \le \Gamma^2/2 \tag{15}$$

then $\dot{F} \le 0$. Moreover, in this case the condition of theorem 2 is obviously satisfied.

Further, the quadratic form (14) of x and $\dot{x}$ is non-trivial (irrespective of the presence of the operator $D \ge 0$): $i^+ = i^- = k$. Consequently, according to Theorem 2, when condition (15) is satisfied, the degree of instability of the state of equilibrium will be equal to k. We recall that (15) is the well-known condition of the impossibility of gyroscopic stabilization, obtained by Hagedorn when there are no dissipative forces.

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